Chapter 5

Stability

Finite Difference Method

Second Session Contents:

Von Neumann Method

John von Neumann

Born: December 28, 1903, Budapest, Hungary

Died: February 8, 1957, Washington, D.C., United States

Education: ETH Zurich

John von Neumann was a Hungarian and later American pure and applied mathematician, physicist, inventor, polymath, and polyglot. He made major contributions to a number of fields, including mathematics, physics, economics, computing, and statistics.

Von Neumann Method

$$u_t - \alpha u_{xx} = 0, \qquad -\infty < x < +\infty, \quad t > 0$$

$$u(x,0) = f(x), \qquad -\infty < x < +\infty$$

Using Fourier series for u function

$$\left.\begin{aligned} u(x,t) &= \hat{u}(t)e^{imx} \\ u(j\Delta x, n\Delta t) &= \hat{u}_n e^{imx_j} \end{aligned}\right\} \xrightarrow{\text{FTCS method}} u_j^{n+1} = u_j^n + r(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

Which $\longrightarrow r = k\alpha/h^2$

$$\longrightarrow \hat{u}_{n+1}e^{imx_j} = \hat{u}_n e^{imx_j} + r\hat{u}_n(e^{imx_{j+1}} - 2e^{imx_j} + e^{imx_{j-1}})$$

Von Neumann Method

$$\xrightarrow{\text{By dividing on } e^{imx_j}} \begin{cases} \hat{u}_{n+1} = \hat{u}_n[1 + r(e^{imh} + e^{-imh} - 2)] \\ \text{Which} \quad h = x_{j+1} - x_j = x_j - x_{j-1} \end{cases}$$

$$\longrightarrow \hat{u}_{n+1} = \hat{u}_n[1 + r(2\cos(mh) - 2)]$$

Trigonometric equations $\quad \hat{u}_{n+1} = \hat{u}_n[1 - 4r\sin^2(mh/2)]$

Amplification factor: $\quad g = \dfrac{\hat{u}_{n+۱}}{\hat{u}_n}$

In this example: $\quad g = ۱ - ۴r\sin^{۲}(mh/۲)$

Von Neumann Method

Stability condition: $\left|\frac{\hat{u}_{n+1}}{\hat{u}_n}\right| \le 1$

For the last example: $|1-4r\sin^2(mh/2)| \le 1 \longrightarrow -2 \le -4r \Longrightarrow r \le \frac{1}{2}$

The important steps of Von Neumann analysis:

– The solution of finite difference problem can be assume as the combination of Fourier modes

$\hat{u}_n e^{imx_j}$

– Using $\hat{u}_n e^{imx_j}$ n finite difference equation and finding $\hat{u}_{n+1}/\hat{u}_n$

– Von Neumann stability condition: $\left|\frac{\hat{u}_{n+1}}{\hat{u}_n}\right| \le 1$ for all modes

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Von Neumann Method

Applications and limitations of Von Neumann method

– Can be used only for linear equations

– The effect of boundary conditions are not considered in stability analysis

– For PDEs discretization which used two time steps, the stability conditions can be determined by:

a) if g is a real number: $|g| \le 1$

b) If g is a complex number: $|g|^2 \le 1$

– For PDEs discretization which used three time steps, the Amplification factor is a matrix. for Eigenvalues of this matrix:

a) if λ_i is a real number: $|\lambda_i| \le 1$

b) If λ_i is a complex number: $|\lambda_i|^2 \le 1$

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Von Neumann Method

– Can be used for several dimensional equations

– Can be used for system of linear equations (in this situation, we should consider the maximum amount of eigenvalues as stability criterion)

– Using graphical solutions for stability analysis (in the situations that calculating the Amplification factor is hard)

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Von Neumann Method

Example 1: Applying Von Neumann criteria for BTCS method

PDE: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \xrightarrow{\text{BTCS method}} u_j^{n+1} = u_j^n + r(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1})$

$u_j^n = \hat{u}_n e^{imx_j}$

$\hat{u}_{n+1} = \hat{u}_n + r\hat{u}_{n+1}(e^{-imh} - 2 + e^{imh})$

$g = 1 + rg(2\cos mh - 2) = 1 - 4rg\sin^2\frac{mh}{2} \longrightarrow g = \frac{1}{1+4r\sin^2\frac{mh}{2}}$

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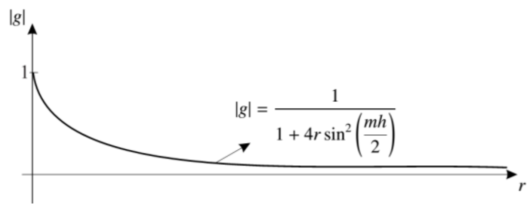
Von Neumann Method

Example 1: Applying Von Neumann criteria for BTCS method



Von Neumann Method

Example 2: Applying Von Neumann criteria for CTCS method

PDE: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \xrightarrow{\text{CTCS method}} u_j^{n+1} = u_j^{n-1} + 2r(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$

$$u_j^n = \hat{u}_n e^{imx_j}$$

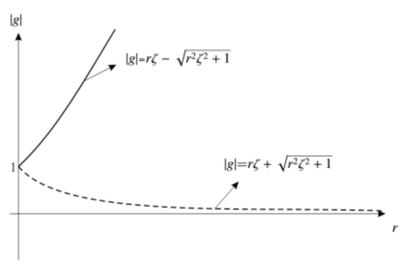
$$\hat{u}_{n+1} = \hat{u}_{n-1} + 2r\hat{u}_n(e^{-imh} - 2 + e^{imh})$$

$$g = \frac{1}{g} + 2r(2\cos mh - 2) \longrightarrow g = 2r(\cos mh - 1) \pm 2\sqrt{r^2(\cos mh - 1)^2 + 1}$$

$$\xrightarrow{\xi = 2(\cos mh - 1)} \quad g = 2r\xi \pm 2\sqrt{r^2\xi^2 + 1}$$

Von Neumann Method

Example 2: Applying Von Neumann criteria for CTCS method



Von Neumann Method

Example 3: Applying Von Neumann criteria for Dufort Frankel method

PDE: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \xrightarrow{\text{Dufort Frankel}} u_j^{n+1} = \left(\frac{1-2r}{1+2r}\right)u_j^{n-1} + \frac{2r}{1+2r}(u_{j+1}^n + u_{j-1}^n)$

$$u_j^n = \hat{u}_n e^{imx_j}$$

$$\hat{u}_{n+1} = \left(\frac{1-2r}{1+2r}\right)\hat{u}_{n-1} + \frac{2r}{1+2r}\hat{u}_n(e^{imh} + e^{-imh})$$

$$g^2 - \left[\frac{4r}{1+2r}\cos mh\right]g - \frac{1-2r}{1+2r} = 0 \longrightarrow g = \frac{2r\cos mh \pm \sqrt{1 - 4r^2\sin^2 mh}}{1+2r}$$

Von Neumann Method

Example 3: Applying Von Neumann criteria for Dufort Frankel method

For small amount of ΔT:

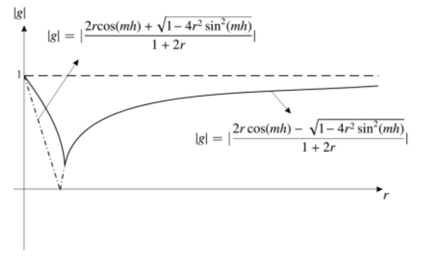
$$4r^2 \sin^2 mh \leq 1 \quad \longrightarrow \quad |g| < 1$$

For large amount of ΔT:

$$4r^2 \sin^2 mh > 1 \quad \longrightarrow \quad |g| = \frac{1-2r}{1+2r}$$ which always is less than one

Von Neumann method

Example 3: Applying Von Neumann criteria for Dufort Frankel method



Matrix Method for Stability Analysis

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1 \quad , \quad 0 \leq t \leq T$$

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < 1$$

Using explicit finite difference method:

$$u_i^n = ru_{i-1}^{n-1} + (1-2r)u_i^{n-1} + ru_{i+1}^{n-1}$$

$$u_0 = u_N = 0$$

$$i = 1, 2, \ldots, N-1$$

Matrix Method for Stability Analysis

$$\begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \end{bmatrix} = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & \ddots \\ & & & r & 1-2r \end{bmatrix} \begin{bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ u_3^{n-1} \\ \vdots \\ u_{N-1}^{n-1} \end{bmatrix}$$

Or $\vec{u}^n = A\vec{u}^{n-1}$ which $\vec{u}$ column matrix and A is $(N-1)\times(N-1)$ rix

$$\vec{u}^n = A\vec{u}^{n-1} = A(A\vec{u}^{n-2}) = \cdots = A^n(\vec{u}^\circ)$$

Which $\vec{u}^\circ = \{f_1, f_2, \ldots, f_{N-1}\}^T$ initial value vector

Matrix Method for Stability Analysis

If error is defined at each point along x axis at t=0

Replacing $\vec{u}^\circ$ by $(\vec{u}^\circ)^*$

$$(\vec{u}^1)^* = A(\vec{u}^\circ)^*, \quad (\vec{u}^2)^* = A(\vec{u}^1)^* = A^2(\vec{u}^\circ)^*$$

And ...

$$(\vec{u}^n)^* = A^n(\vec{u}^0)^*$$

Error vector can be defined as:

$$\vec{e} = \vec{u} - \vec{u}^*$$

Therefore:

$$\vec{e}^n = \vec{u}^n - (\vec{u}^n)^* = A^n(\vec{u}^\circ - (\vec{u}^\circ)^*) = A^n\vec{e}^\circ$$

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Matrix Method for Stability Analysis

$$\vec{e}^n = \vec{u}^n - (\vec{u}^n)^* = A^n(\vec{u}^\circ - (\vec{u}^\circ)^*) = A^n\vec{e}^\circ$$

The error propagation formula is the same as u

- ❖ Based on superposition principle in linear problems, we can only study the behavior of one error
- ❖ A finite-difference method is stable if $\underset{n \to \infty}{\vec{e}^n}$ =limited values
- ❖ It was shown that matrix A has N-1 eigenvector. Therefore, the error vector can be determined by N-1 eigenvector as follow

$$\vec{e}^\circ = \sum_{k=1}^{N-1} C_k \vec{V}_k$$

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Matrix Method for Stability Analysis

$$\vec{e}^1 = A\vec{e}^\circ = A\sum C_k\vec{V}_k = \sum C_k A\vec{V}_k \qquad A\vec{V}_k = \lambda_k\vec{V}_k$$

$$\vec{e}^1 = \sum C_k\lambda_k\vec{V}_k$$

$$\vec{e}^2 = \sum C_k\lambda_k^2\vec{V}_k$$

$$\vec{e}^n = \sum C_k\lambda_k^n\vec{V}_k$$

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Matrix Method for Stability Analysis

$$\vec{e}^1 = A\vec{e}^\circ = A\sum C_k\vec{V}_k = \sum C_k A\vec{V}_k \qquad A\vec{V}_k = \lambda_k\vec{V}_k$$

$$\vec{e}^1 = \sum C_k\lambda_k\vec{V}_k$$

$$\vec{e}^2 = \sum C_k\lambda_k^2\vec{V}_k$$

$$\vec{e}^n = \sum C_k\boxed{\lambda_k^n}\vec{V}_k \longrightarrow \lambda_k = 1 - 4r\sin^2\frac{k\pi}{2N}$$

$$\left|1 - 4r\sin^2\frac{k\pi}{2N}\right| \le 1 \longrightarrow r \le \frac{1}{2}$$

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Example

$$u_i^{n+1} = \boldsymbol{A} u_i^n \qquad A = \begin{pmatrix} 1+r & r & 0 \\ 0 & 1+r & r \\ -r & 0 & 1+r \end{pmatrix} \qquad r = \Delta t / \Delta x$$

$$|\boldsymbol{A} - \lambda I| = 0$$

$$(1 + r - \lambda)^3 - r^3 = 0 \qquad \begin{cases} \lambda_1 = 1 \\ \lambda_{2,3} = \dfrac{(2+3r) \pm ir\sqrt{3}}{2} \end{cases}$$

$|\lambda| > 1$ ⟶ The method is not stable

for any r

Stability Analysis of Crank Nicolson Method

$$\begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} (2-2r) & r & & & \\ r & (2-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (2-2r) & r \\ & & & r & (2-2r) \end{bmatrix} \begin{bmatrix} u_1^{n} \\ u_2^{n} \\ \vdots \\ \vdots \\ u_{N-1}^{n} \end{bmatrix}$$

Stability Analysis of Crank Nicolson Method

$$\begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} (2-2r) & r & & & \\ r & (2-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (2-2r) & r \\ & & & r & (2-2r) \end{bmatrix} \begin{bmatrix} u_1^{n} \\ u_2^{n} \\ \vdots \\ \vdots \\ u_{N-1}^{n} \end{bmatrix}$$

$$(2I - rB)u^{n+1} = (2I + rB)u^n$$

$$u^{n+1} = (2I - rB)^{-1}(2I + rB)u^n$$

$$\boldsymbol{B} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

Stability Analysis of Crank Nicolson Method

$$u^{n+1} = \boxed{(2I - rB)^{-1}(2I + rB)}\,u^n$$

$$(2I - rB)^{-1}(2I + rB) = A$$

$$\lambda_k = \frac{2 - 4r\sin^2\left(\dfrac{k\pi}{2N}\right)}{2 + 4r\sin^2\left(\dfrac{k\pi}{2N}\right)}, \qquad (k = 1, 2, \ldots, N-1)$$

$|\lambda_k| < 1$ ⟶ The method is unconditionally stable

for any r

Example

$$(2I - rB)u^{n+1} = (2I + rB)u^n = \{4I - (2I - rB)\}u^n$$

$$Cu^{n+1} = (4I - C)u^n$$

$$u^{n+1} = (4C^{-1} - I)u^n$$

$$C = \begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix}$$

Example

$$(2I - rB)u^{n+1} = (2I + rB)u^n = \{4I - (2I - rB)\}u^n$$

$$Cu^{n+1} = (4I - C)u^n$$

$$u^{n+1} = \boxed{(4C^{-1} - I)}u^n$$

$$C = \begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix}$$

$$\left|\frac{4}{\lambda} - 1\right| \le 1$$

$$\lambda \ge 2$$

Appendix $\quad |\lambda - 2 - 2r| \le 2r$

$$-2r \le \lambda - 2 - 2r \le 2r$$

$$2 \le \lambda \le 2 + 4r \longrightarrow \boxed{\text{unconditionally stable}}$$

Stability Analysis of Neumann Boundary Condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1$$

$$\frac{\partial u}{\partial x} = \begin{cases} h_1(u - v_1) & x = 0 \quad t \ge 0 \\ h_2(v_2 - u) & x = 1 \quad t \ge 0 \end{cases}$$

$h_1, h_2, v_1, v_2 = const$
$h_1, h_2 > 0$

$$\frac{u_1^j - u_{-1}^j}{2\Delta x} = h_1(u_\circ^j - v_1)$$ Boundary condition

$$\frac{u_{N+1}^j - u_{N-1}^j}{2\Delta x} = -h_2(u_N^j - v_2) \quad (N\Delta x = 1)$$

$$u_i^{j+1} = ru_{i-1}^j + (1 - 2r)u_i^j + ru_{i+1}^j$$

Stability Analysis of Neumann Boundary Condition

$$u_i^{j+1} = ru_{i-1}^j + (1 - 2r)u_i^j + ru_{i+1}^j$$

By omitting u_{N+1}^j and u_{-1}^j

$$\begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ \vdots \\ u_N^{j+1} \end{bmatrix} = \begin{bmatrix} \{1-2r(1+h_1\Delta x)\} & 2r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & 2r & \{1-2r(1+h_2\Delta x)\} \end{bmatrix} \times \begin{bmatrix} u_0^j \\ u_1^j \\ \vdots \\ \vdots \\ u_N^j \end{bmatrix} + \begin{bmatrix} 2rh_1v_1\Delta x \\ 0 \\ \vdots \\ \vdots \\ 2rh_2v_2\Delta x \end{bmatrix}$$

Stability Analysis of Neumann Boundary Condition

$$u_i^{j+1} = ru_{i-1}^j + (1-2r)u_i^j + ru_{i+1}^j$$

By omitting u_{N+1}^j and u_{-1}^j

$$\begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ \vdots \\ u_N^{j+1} \end{bmatrix} = \boxed{\begin{bmatrix} \{1-2r(1+h_1\Delta x)\} & 2r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & 2r & \{1-2r(1+h_2\Delta x)\} \end{bmatrix}} \times \begin{bmatrix} u_0^j \\ u_1^j \\ \vdots \\ \vdots \\ u_N^j \end{bmatrix} + \begin{bmatrix} 2rh_1v_1\Delta x \\ 0 \\ \vdots \\ \vdots \\ 2rh_2v_2\Delta x \end{bmatrix}$$

This matrix determines the error propagation

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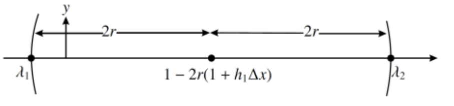
Stability Analysis of Neumann Boundary Condition

Based on Brauer Theorem

$$|\lambda - \{1-2r(1+h_1\Delta x)\}| \le 2r$$



$$\lambda_1 = 1 - 2r(2 + h_1\Delta x) \qquad \lambda_2 = 1 - 2rh_1\Delta x$$

$$|\lambda_1| \le 1 \qquad |\lambda_2| \le 1$$

$$r \le \frac{1}{2 + h_1\Delta x} \qquad r \le \frac{1}{h_1\Delta x}$$

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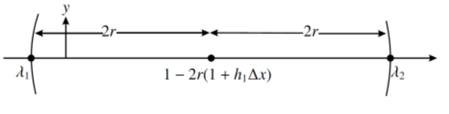
Stability Analysis of Neumann Boundary Condition

Based on Brauer Theorem

$$|\lambda - \{1-2r(1+h_1\Delta x)\}| \le 2r$$



$$\lambda_1 = 1 - 2r(2 + h_1\Delta x) \qquad \lambda_2 = 1 - 2rh_1\Delta x$$

$$|\lambda_1| \le 1 \qquad |\lambda_2| \le 1$$

$$\boxed{r \le \frac{1}{2 + h_1\Delta x}} \qquad r \le \frac{1}{h_1\Delta x}$$

Minimum value

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Stability Analysis of Neumann Boundary Condition

$$r \le \frac{1}{2 + h_1\Delta x} \qquad r \le \frac{1}{2 + h_2\Delta x} \qquad r \le \min\left\{\frac{1}{2 + h_1\Delta x}\,;\,\frac{1}{2 + h_2\Delta x}\right\}$$

$$e_{j+1} = (4B^{-1} - I)e_j$$

$$B = \begin{bmatrix} 2+2r(1+h_1\Delta x) & -2r & & & \\ -r & 2+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 2+2r & -r \\ & & & -2r & 2+2r(1+h_2\Delta x) \end{bmatrix}$$

$$|\lambda - \{2+2r(1+h_1\Delta x)\}| \le 2r$$

$$2 + 2rh_1\Delta x \le \lambda$$

$$\lambda \ge 2$$

$$\lambda \ge 2 + 2rh_2\Delta x$$

unconditionally stable

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